22

DYNAMICS OF VIBRATION OF A CANTILEVER UNDER LATERAL IMPACT OF AN ELASTIC LOAD. PART III. (GENERAL THEORY)

B. B. BANERJEE

(*Received September* 11, 1965)

ABSTRACT. In part II, the general expression for the displacement of any point of the cantilever, under lateral impact of a load is given. In this paper, the general expression for the pressure exerted by the load on the cantilever is worked out. The duration of contact is the lowest positive root of t, other than zero, obtained by solving Pressure $P = 0$, for the given struck point. This duration of contact can be directly obtained from the expression of pressure. Existence of different modes of vibration governs the magnitude of the duration of contact. Pressure is found to be directly proportional to the striking velocity and depends on all physical constants as appear in the general displacement equation, given in Part II. It has been possible to explain the occurrence of the phenomenon of multiple contacts from the pressure-time and the velocity—time relations. The extension of the general theory in light of Hertz's theory of impact explains fully the dependence of duration of contact on the impinging velocity of the hammer.

INTRODUCTION

In Part II the general expression for the displacement of any point of the cantilever (including the struck point) is given. In this paper using the same operational method due to Heaviside, the general expression for pressure of impact is worked out. The duration of contact is directly obtained from the pressure equation. Without entering into lengthy mathematical computation to explain the occurrence of the phenomenon of multiple contacts, it has been possible to explain the higher contacts in light of present theory. In previous theories, the physical aspect of the problem was not so much tackled.

PRESSURE OF IMPACT

Pressure exerted by the hammer is given by eqn. (6.1), (Banerjee, 1966).

$$P = m\frac{d^2z}{dt^2} = -E_2 u$$

where

$$z = y_a + u$$

and

$$u = -\frac{E_1 I}{E_2} y_a n^3 f(D) \text{ from eqn. (9.2) (Banerjee, 1966).}$$

$$y_a = \frac{F(D)}{F_1(D)} \cdot v_0 \text{ from eqn. (9.1) (Banerjee, 1966).}$$

Therefore,

$$P = -mv_0 \frac{F(D)}{F_4(D)}$$

Where $F(D)$ stands for D and

$$F_4(D) = 1 + \frac{m}{E_2} D^2 + \gamma \frac{m}{M} \frac{\begin{array}{l} +\cosh k_2\gamma \sin k_2\gamma + \sinh k_1\gamma \cos k_1\gamma \\ +\cos k_1\gamma \cos k_2\gamma \sinh \gamma - \sinh k_2\gamma \cos k_2\gamma \\ -\cosh k_1\gamma \sin k_1\gamma - \cosh k_1\gamma \cosh k_2\gamma \sin \gamma \end{array}}{2(1+\cos\gamma \cosh \gamma)} \quad \ldots (1)$$

Therefore with the help of Heaviside's expansion theorem

$$P = -mv_0\left[\frac{F(0)}{F_4(0)} + \sum \frac{F(\alpha_s)}{\alpha_s, F_4'(\alpha_s)} e^{\alpha_s t}\right] \quad \ldots (2)$$

But putting $D = 0$, we get $F(0) = 0$, and $F_4(0) = 1$.
and so

$$\frac{F(0)}{F_4(0)} = 0$$

For roots of D from $F_4(D) = 0$, we have $F_4(D) = 0$, whence

$$\frac{2(1+\cosh\gamma \cos\gamma)}{\begin{array}{l}\sinh k_2\gamma \cos k_2\gamma + \cosh k_1\gamma \sin k_1\gamma + \cosh k_1\gamma \cosh \\ k_2\gamma \sin\gamma - \cosh k_2\gamma \sin k_2\gamma - \sinh k_1\gamma \cos k_1\gamma \\ -\cos k_1\gamma \cos k_2\gamma \sin\gamma\end{array}} = \frac{\frac{m}{M}\gamma}{1 - \frac{E_1 I}{E_2} \cdot \frac{m}{M} \cdot \frac{\gamma^4}{l^3}} \quad \ldots (3)$$

Equations (3) and (7) (Banerjee, 66) are same and so the roots of γ as obtained from eqn. (24), in case of displacement in Part II (Banerjee, 1966) and eqn. (3) are having same set of values

Thus

$$nl = \gamma_s, \ (s = 1, 2, 3, 4, \ldots r)$$
$$D = [\alpha_s] = \pm i q_s$$
$$q_s = \gamma_s^2 \cdot \sqrt{\frac{E_1 I}{M l^3}}$$

and

$$F_4'(\alpha_s) = \frac{1}{2\alpha_s}\left\{\frac{3m}{E_2}\alpha_s^2 - 1 + \gamma_s\left(1 + \frac{m}{E_2}\alpha_s^2\right) \frac{\sinh\gamma_s \cos\gamma_s - \cosh\gamma_s \sin\gamma_s}{1+\cosh\gamma_s \cos\gamma_s}\right.$$
$$+2[k_2 \sinh k_2\gamma_s \sin k_2\gamma_s - k_1 \sinh k_1\gamma_s \sin k_1\gamma_s] + \cosh\gamma_s \cos k_1\gamma_s \cos k_2\gamma_s$$
$$-\cosh k_1\gamma_s \cosh k_2\gamma_s \cos\gamma_s - k_1[\sinh\gamma_s \sin k_1\gamma_s \cos k_2\gamma_s + \sinh k_1\gamma_s$$
$$\left. + \frac{m}{M}\gamma_s^2 \frac{\cosh k_2\gamma_s \sin\gamma_s - k_2[\sinh\gamma_s \sin k_2\gamma_s \cos k_1\gamma_s + \sin\gamma_s \sinh k_2\gamma_s \cosh k_1\gamma_s]}{2(1+\cosh\gamma_s \cos\gamma_s)}\right\} \quad (4)$$

After simplification we get, with the help of Heaviside's expansion theorem, the pressure exerted by the load as

$$P = -4mv_0 \Sigma\, B_s q_s \sin q_s t \qquad \ldots \quad (5)$$

where

$$\frac{1}{B_s} = \Bigg\{ 1 + \frac{3m}{E_2}\, q_s^2 + \gamma_s \left[1 - \frac{m}{E_2}\, q_s^2 \right] \frac{\cosh \gamma_s \sin \gamma_s - \sinh \gamma_s \cos \gamma_s}{1 + \cosh \gamma_s \cosh \gamma_s}$$

$$\begin{aligned}
&2[k_1 \sinh k_1\gamma_s \sin k_1\gamma_s - k_2 \sinh k_2\gamma_s \sin k_2\gamma_s) \\
&+ \cosh k_1\gamma_s \cosh k_2\gamma_s \cos \gamma_s - \cosh \gamma_s \cos k_1\gamma_s \cos k_2\gamma_s \\
&+ k_1[\sinh \gamma_s \sin k_1\gamma_s \cos k_2\gamma_s + \sinh k_1\gamma_s \cosh k_2\gamma_s \sin \gamma_s]
\end{aligned}$$

$$+ \frac{m}{M}\gamma_s^2 \; \frac{+k_2[\sinh \gamma_s \sin k_2\gamma_s \cos k_1\gamma_s + \sinh k_2\gamma_s + \sinh k_2\gamma_s \cosh k_1\gamma_s \sin \gamma_s}{2(1 + \cosh \gamma_s \cos \gamma_s)} \Bigg\}$$

Thus the pressure exerted by the hammer during impact is directly proportional to the striking velocity and it involves all the physical constants appearing in the displacement equation (12.1), (Banerjee 1966) for the struck point

DURATION OF IMPACT

The duration of contact which plays a very important part in the dynamics of vibration of a cantilever under lateral impact of an elastic load, is the time, measured from the instant the load comes in contact with the bar and maintains its contact till it separates out. There are cases where the phenomenon of multiple contact arises. The hammer or the bar may again overtake each other with a short period of separation and the hammer continues to remain in contact till another separation occurs. This overtaking may take place several times with interving separations of short intervals until the load is able to leave the region of vibration of the bar. In such cases, the total duration of impact may be the time that elapses between the instant the impact begins and the instant, the hammer finally breaks off with the bar Thus the duration of impact is obviously the lowest positive root of t, other than zero, obtained by solving $P = 0$, for the given struck point, in the pressure equation (5) This time of collision is used in the expression for velocity (eqn. 13, Banerjee 1966) to obtain the velocity of the hammer at the end of contact. Drawing the pressure-time curve as per equation (5) for pressure with required number of terms of the series we may obtain directly the magnitude of the duration of impact in specific case, of a beam hammer system.

Existence of different modes of vibration of the bar, as given by different terms of the displacement series (eqn. 12.1) (Banerjee 1966) governs the magnitude of the duration of contact. The experiments reported by the writer help to ascertain the number of terms to be included in the pressure equation (5) in order to calculate the exact value of the duration of impact. Young's rule also holds good That is if the position of any node of a particular mode of vibration is the struck point, the mode is not excited, and it is found under this condition, that in our pressure equation (5), the terms corresponding to that mode vanishes. Thereforeit is possible to easily obtain the duration of contact, using required number of terms in the expression for pressure (eqn 5) of the present theory which previous theories failed to do

MULTIPLE CONTACTS

We proceeded to show that the load and the bar being attached during impact, move with a common velocity (eqn. 13) (Banerjee 1966) But at the instant $P = 0$, in our pressure eqn (5), the contact terminates. The motion of the bar is affected due to detachment of a moving mass and so its equation of motion is modified after impact cases The change in the velocity of the bar is in opposite sense to that suffered by the load at the instant separation ensues This will create a space gap between the load and the bar At the end of a contact (when $P = 0$), the velocity may have any direction with respect to positive impinging velocity of the load and is determined by the striking distance and the number of modes stimulated into activity during impact When this velocity, at the enp of a contact comes out as positive, the load moves in a forward direction with a uniform velocity as it has no resistance to overcome after impact ceases, and shall meet the cantilever again which would be vibrating with the modified displacement law obtainm at the beginning of separation with load. Thus the second contact in such cases is a fresh contact, compression of the load follows and pressure is built up again to reduce to zero, when another separation occurs Phenomenon of multiple contacts mab e observed in the region, the load reverses its direction where the magnitude of the velocity of the bar shifts towards higher value So there shall be a sudden set back in the velocity of the bar due to any detachment of the moving load Overtaking the hammer to make another contact in this region depends on the recovery of this time-lag by the bar by suitably modifying its displacement condition Thus at the final termination of contact three conditions are satisfied, namely, the pressure between the load and the cantilever is zero. the velocity of the load is negative and the displacement-time curves of the bar and the load do not meet again The higher contacts other than the first are therefore different from what is given by our general dynamics using pressure equation (5). The higher contacts shall slightly modify the vibration form of the contilever obtained from the displacement equation (12.1) (Banerjee 1966).

From the general expression for pressure (eqn. 5), is found that the duration of impact is not altered by the impinging velocity But consizering the effects of elastic collisions of two bodies. Hertzian impact is superimposed over the general condition of elastic impact, dealt in the previous parts

APPLOCATION OF HERTZ'S THEORY OF IMPACT

Ghosh (1940) (in case of struck string) has considered the duration of contact to be divided into three distinct periods In our problem too, we follow the same arguments. Thus the three periods are named as 'First Hertz', 'Hooke' and Second Hertz' periods respectively During the First Hertz period, the displacement of the bar in our case, is not appreciable, and the pressure of impact obeys Hertz's law of impact uhtil a certain pressure is developed to make the displacement of the cantilever appreciable As soon as the bar was displaced, the pressure obeys 'Hooke's law and the motion of the hammer is given by displacement eqn. (12 1) (Banerjee 1966) of the present general theory The duration of contact during this period is calculated by equating the pressure equation (5) to zero as usual After this 'Hooke' period is over, the extra compression developed in this period is released and the 'Second Hertz' period begins It is assumed that the 'First Hertz' and the 'Second Hertz' periods have the same durations τ (say) Therefore the total duration of contact is $\phi_0+2\tau$, where ϕ_0 is the duration of Hooke's period, as calculated from our general expression of pressure

Following the method of Ghosh (1940), we write,

The pressure exerted due to Hertzian impact

$$m\frac{d^2z}{dt^2} = m\frac{d^2y_a}{dt^2}+m\frac{d^2u}{dt^2} \quad \dots \quad (6.1)$$

$$= -eu^{3/2} \quad \dots \quad (6.2)$$

$$z = y_a+u, \text{ and let } n_1 = \frac{4}{5}\frac{e}{m}, \quad \dots \quad (6.3)$$

where e is a constant depending on the geometric and elastic properties of the bar and the hammer as also their shapes at the contact surface Assuming $y_a = 0$, during Hertz's periods, and $v_0 = \dfrac{du}{dt}$, at $t = 0$, the time to produce compression u_0, where u_0 is the limiting value of u at the end of a 'Hertz' period is given by Ghosh (1940) as

$$\tau = \frac{u_0}{v_0}\left[1+\frac{n_1u^{5/2}}{7v_0^2}+\dots\right] \quad \dots \quad (7)$$

Thus taking the first approximation, i e rataining only the first term

$$\phi = \phi_0+2\tau \text{ or } \phi-\phi_0 = \frac{2u_0}{v_0} \quad \dots \quad (8)$$

ϕ_0 is calculated in the usual way from pressure function (eqn 5). The value of u_0 is obtained by selection from a particular set of experimental values of the duration of contact.

ACKNOWLEDGEMENT

My best thanks are due to Prof. M. M. Ghosh, D.Sc., vice-principal, City College Calcutta for his helpful guidance.

REFERENCES

Banerjee, B B., 1966, *Indian J Phys*, **40**, 198, 208.
Ghosh, M. 1940, *Indian J Phys*. **14**. 495.